Note 192: Subsentential Syntax

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I find Bob's recent notes on the construction of subsentential syntactical structure out of base consequence relations among sets of sentences very attractive. A thoroughgoing use-theoretic inferentialism needs a way to make sense of subsentential structure in terms of the norms that govern the uses of sentences in inferences.

I want to explore a similar but distinct strategy for doing this. The main reason for this is that I worry that the idea that two sentences share a predicate iff they can be substituted *salva consequentia* isn't want we want. This is for two reasons:

1. Two sentences are everywhere substitutable *salva consequentia* in an implication-space model iff they have the same RSR in that model iff they express the same content in that model. So what we get from this is the idea of synonymous sentences and I doubt that this idea is rich enough for what we want.

2. The sentences that share a predicate might be synonymous if the singular terms in them are coreferential, which would be ensured by an identity statement. But I am not sure that I understand how the ideas can be used to make sense of when sentences share predicates or singular terms when no identity statements are assumed.

Probably I am just missing something in Bob's construction. In any event, it motivated me to think a bit about the matter myself. Here are some initial ideas. It is all very rough; and Bob's note is more important for tomorrow. But I thought I

1 Basic Ideas

We must use some information in our material base. Suppose for simplicity that the possible syntactic structures that we want to recover are predicate-name(s) structures (no variables, etc.). If we let predicates and names carry information about inferential goodness separately, I don't see how this could possibly work. To see this, attribute any syntactic structure you like to the sentences of a material base, and then ask yourself whether you have any reason to think that this is not the "real" syntactic structure. I cannot see what reason you could possible have for a negative answer to that question. The space of possibilities seems completely unconstrained to me.

One solution that I am willing to consider is this: Stipulate that identities of individuals don't matter for inferential goodness. I take this to be something similar to what Wittgenstein means in the TLP when he says that objects are colorless.

Assumption 1. Objects are colorless*: If* $\Gamma \mid\!\sim \Delta$ *and* Γ' *and* Δ' *are results of bijections of the singular terms of the language to themselves (so permutations of the names of the language), then* $\Gamma' \mid\!\sim \Delta'$.

If we could now find ways in which we could map the sentences of our language onto themselves in such a way that the consequence relation is preserved, we could say that this mapping is, in effect, a permutation of the names of the language.

Definition 2. *n-folding*: An n-folding of a material base, $F_n(\mathfrak{B})$, is an collection of $n-1$ nontrivial automorphisms (preserving consequence as the only relevant structure) of the sentences of the language, $\mathfrak{L}$,[1] of the base such that $Aut^i_{\mathfrak{L}} \in F_n(\mathfrak{B})$ iff (i) $\Gamma \mid\!\sim \Delta \in \mathfrak{B}$ iff $Aut^i_{\mathfrak{L}}(\Gamma) \mid\!\sim Aut^i_{\mathfrak{L}}(\Delta) \in \mathfrak{B}$ and (ii) if $Aut^i_{\mathfrak{L}} \in F_n(\mathfrak{B})$ and $Aut^j_{\mathfrak{L}} \in F_n(\mathfrak{B})$ and $Aut^i_{\mathfrak{L}}(A) = B$ and $Aut^j_{\mathfrak{L}}(A) = C$, then $\exists Aut^k_{\mathfrak{L}} \in F_n(\mathfrak{B})$ such that $Aut^k_{\mathfrak{L}}(B) = C$.

The idea here is that there are symmetries in our base, i.e., we can "rotate" or "fold" our base in several ways that yield a result that is indistinguishable from the original base. The sentences that are mapped onto each other by such "rotations" or "foldings" are substitution instances of the same schema of an atomic sentence, i.e., substitution instances of the same predicate.

The second condition is there to ensure that we get classes of sentences such that all automorphisms in an n-folding map sentences in a particular class always only to other sentences in that same class. The idea is that these classes are sentences that share their predicate, i.e., they are substitution variants of each other.

Definition 3. *Substitution variants*: Sentences A and B in $\mathfrak{L}$ are substitution variants of each other under n-folding, $F_n(\mathfrak{B})$, iff $\exists Aut^i_{\mathfrak{L}} \in F_n(\mathfrak{B})$ such that $Aut^i_{\mathfrak{L}}(A) = B$.

Notice that our n-foldings never map two atomic sentences to the same atomic sentence. For, automorphism are isomorphisms and hence (one-to-one and onto). So, if we think of n-foldings as—behind the curtain of the subsentential structure really being—permutations of singular terms, then they never map two singular terms to the same singular term (if we think we already see the underlying structure). For that would lead to two sentences being mapped to the same sentence (e.g. if a and b and mapped to c, then Ga and Gb and both mapped to Gc). So, identity of objects is encoded by identity of terms (at this fundamental level).

[1] Notice that the base isn't strictly speaking sufficient to determine the language because there might be sentences that figure only in bad inferences. So, we really need to either specify the language independently, or include the complement of the base at the bottom of our construction.

Notice also automorphisms can be composed, and there can be basis of a set of automorphisms in the following sense:

Definition 4. *Basis of n-folding*: The set $\{Aut^1_{\mathfrak{L}}, ..., Aut^j_{\mathfrak{L}}\}$ is a basis of an n-folding $F_n(\mathfrak{B})$ iff $\{Aut^1_{\mathfrak{L}}, ..., Aut^j_{\mathfrak{L}}\} \subseteq F_n(\mathfrak{B})$ and, for every $Aut^i_{\mathfrak{L}} \in F_n(\mathfrak{B})$, there is a sequence of a subset of $\{Aut^1_{\mathfrak{L}}, ..., Aut^j_{\mathfrak{L}}\}$ such that $Aut^i_{\mathfrak{L}}$ is the same as applying the sequence of automorphisms in the subset (in that order).

To see why the notion of a basis might come in handy, notice that you can produce all permutations of singular terms of a language by chaining together swaps of just two singular terms. So swaps of just two singular terms form a basis for all permutations of singular terms. Since we will think of our automorphisms as permutations of singular terms, we should expect there to be at least one basis that corresponds to swaps of two singular terms.

Summar: The basic ideas then are the following.

- Subsentential structure is always relative to an n-folding, which is a symmetry in the base. Since there might be many symmetries in a base, there might be many equally correct but incompatible analyses of the subsentential structure of the sentences of a base.

- Relative to an n-folding of a base, predicates are what is shared by sentences that are substitution variants of each other.

- What differs between substitution variants of two sentences is what singular terms occur in what order in them. (However, since we just keep track of how many variants we can create, the identity of the terms doesn't matter.)

- Put differently, permutations of singular terms are axes of symmetry of the base consequence relation.

- Predicates are the classes of points mapped onto each other by these symmetries.

- Attributing to a language subsentential structure is a kind of "dimension-reduction" procedure for base consequence relations. If you know how any substitution instances are related by consequence, then you know how any of the other substitution instances of the same set of sentences are related. This is a way to compress information about the base consequence relation. And what allows one to do this is the symmetry. One folds the base up into a smaller part and an instruction for expanding it; where the instruction for expanding it is the subsentential structure of the sentences.

2 Syntactic Hypotheses

For our language, we need it to be the case that it can be generated by our subsentential syntax. That is, there must be a bijection between $\mathfrak{L}$ and the candidate subsentential syntax that we propose. If, e.g., we say that our language as singular terms a, b, c and on-term predicate F and relational predicate G, then we need to have in $\mathfrak{L}$ exactly one sentence for $Fa, Fb, Fc, Gaa, Gab, Gac, Gba, Gbb, Gbc, Gca, Gcb, Gcc$.

- Call such a mapping a "syntactic hypothesis" for a base.

- Notice that syntactic hypotheses might get complicated if we allowed for more complex typing, functors, etc.

- I am using the simplest kind of syntactic hypotheses because I think I have some idea for how to get (permutations of) singular terms and predicates out of symmetries of base consequence relations. If we wanted to entertain more complex syntactic hypotheses, we would need a more complex story.

- This means that my approach lacks one feature that Bob claims for his, namely the selectivity of predicates. My predicates and singular terms are promiscuous; they combine freely and every combination is a syntactically correct sentence.

3 Questions that n-Foldings Should Answer

It is not clear to me that the definition of n-foldings above is really what we want. Moreover, n-foldings must have more structure that allow us to recover more detailed information about the subsentential structure of our sentences. In particular, we would want our n-foldings to answer the following questions:

1. How (if at all) is the arity of a predicate that is shared by the substitution variants in a class determined?

2. Can we determine which sentences share occurrences of singular terms?

3. What corresponds to the order in which singular terms occur in a sentence?

4. What do atomic sentences without singular terms (like feature-placing sentences) look like?

5. Are their n-foldings that do not support any subsentential analysis of their sentences? That is, is there any n-folding that cannot be explained by any syntactic hypothesis? And, more generally: must we require more than what is in Definition 2 above in order to ensure that we have a structure that delivers answers to the questions just stated?

6. What does it take for an n-folding to answer the questions 1-4 above in a unique way?

7. How is the number of foldings related to the number of singular terms? What about infinite base languages with n-foldings where the n is some infinite cardinality?

I guess that the ideas in the previous sections must be revised in order to yield good answers to these questions. But I am optimistic that this is, in principle, possible.

In order to get a feeling for the lay of the land, let's try to reverse engineer this. So let's try to start with a standard first-order, atomic language, without identity, or functors. We can then see what kind of n-foldings these generate.

4 Example One: Going in the Other Direction

I will take some ridiculously simply languages, and see what natural n-foldings they yield. So here is one:

- Predicates:
  - 0-place: p, q
  - 1-place: G, H
  - 2-place: R
- Singular terms: $a, b, c,$
- Schematic base: all CO-instances, and all instances of:
  1. $G(\alpha), R(\alpha, \beta) \mathrel{|\!\sim} H(\beta)$
  2. $R(\alpha, \beta) \mathrel{|\!\sim} R(\beta, \alpha), q$
  3. $p, H(\alpha) \mathrel{|\!\sim} G(\alpha)$

Given that we have three singular terms, there are six permutations of these terms. And we expect all of them to be axis of symmetry of our base.

That is, they are all automorphisms in an n-folding of our base. This is what we get from the fact that objects are colorless. We get, for instance, all the substitution instances of 1:

- $G(a), R(a,b) \mathrel{|\!\sim} H(b)$; $G(a), R(a,c) \mathrel{|\!\sim} H(c)$; $G(b), R(b,a) \mathrel{|\!\sim} H(a)$; $G(b), R(b,c) \mathrel{|\!\sim} H(c)$; $G(c), R(c,a) \mathrel{|\!\sim} H(a)$; $G(c), R(c,b) \mathrel{|\!\sim} H(b)$; $G(a), R(a,a) \mathrel{|\!\sim} H(a)$; $G(b), R(b,b) \mathrel{|\!\sim} H(b)$; $G(c), R(c,c) \mathrel{|\!\sim} H(c)$.

The identity mapping of the singular terms to themselves yield these same instances of 1. The other permutations of singular terms yield always the same instances, just in different orders:

- $(abc) \mapsto (acb)$:
  $G(a), R(a,c) \mathrel{|\!\sim} H(c)$; $G(a), R(a,b) \mathrel{|\!\sim} H(b)$; $G(c), R(c,a) \mathrel{|\!\sim} H(a)$; $G(c), R(c,b) \mathrel{|\!\sim} H(b)$; $G(b), R(b,a) \mathrel{|\!\sim} H(a)$; $G(b), R(b,c) \mathrel{|\!\sim} H(c)$; $G(a), R(a,a) \mathrel{|\!\sim} H(a)$; $G(c), R(c,c) \mathrel{|\!\sim} H(c)$; $G(b), R(b,b) \mathrel{|\!\sim} H(b)$.

- $(abc) \mapsto (bac)$
  $G(b), R(b,a) \mathrel{|\!\sim} H(a)$; $G(b), R(b,c) \mathrel{|\!\sim} H(c)$; $G(a), R(a,b) \mathrel{|\!\sim} H(b)$; $G(a), R(a,c) \mathrel{|\!\sim} H(c)$; $G(c), R(c,b) \mathrel{|\!\sim} H(b)$; $G(c), R(c,a) \mathrel{|\!\sim} H(a)$; $G(b), R(b,b) \mathrel{|\!\sim} H(b)$; $G(a), R(a,a) \mathrel{|\!\sim} H(a)$; $G(c), R(c,c) \mathrel{|\!\sim} H(c)$.

- $(abc) \mapsto (bca)$
  $G(b), R(b,c) \mathrel{|\!\sim} H(c)$; $G(b), R(b,a) \mathrel{|\!\sim} H(a)$; $G(c), R(c,b) \mathrel{|\!\sim} H(b)$; $G(c), R(c,a) \mathrel{|\!\sim} H(a)$; $G(a), R(a,b) \mathrel{|\!\sim} H(b)$; $G(a), R(a,c) \mathrel{|\!\sim} H(c)$; $G(b), R(b,b) \mathrel{|\!\sim} H(b)$; $G(c), R(c,c) \mathrel{|\!\sim} H(c)$; $G(a), R(a,a) \mathrel{|\!\sim} H(a)$.

- $(abc) \mapsto (cab)$
  ...

- $(abc) \mapsto (cba)$
  ...

It is obvious that the same thing happens with the instances of 2 and 3 and the instances of CO. So, the six permutations of our three singular terms give us a 6-folding of our base relation. For, these mappings are automorphisms on the language preserving the consequence relation, i.e., good implications are mapped to good implications and bad implications are mapped to bad implications. And because we get basically a "kind of" Cartesian product of the permutation instances in each base and the permutations of the base, these automorphisms satisfy the second condition. E.g., $(abc) \mapsto (bac)$ maps $R(a,b)$ to $R(b,a)$; and $(abc) \mapsto (bca)$ maps $R(a,b)$ to $R(b,c)$. So, our second condition requires that one of our automorphisms maps $R(b,a)$ to $R(b,c)$. And, of course, $(abc) \mapsto (cba)$ does that.

Obviously, the automorphisms in our 6-folding always map substitution instances of sentences to each other. So we get five classes of sentences that are mapped each other (which each class), namely from of each of: $p, q, G(\cdot), H(\cdot), R(\cdot, \cdot)$.

So far so good, but we would like to be able to recover more details information from our 6-folding.

5 Towards Some Answers

- Ad 1 above: Is there a way to tell from the 6-folding of what arity the sentences in our classes of substitution items are? Some observations:
  - All permutations of singular terms map p and q to themselves, while all other sentences are mapped to a different sentence by at least one permutation. And it seems to be generally true that sentences of 0-arity are all any only those sentences that are mapped to themselves by every automorphism in an n-folding.
  - For unary predicates, there are three pairs of permutations such that in every class of substitution variants for binary predicates, there are exactly two sentences that are mapped

to each other by the two permutations in these pairs, while the automorphism maps all other sentences in the class to themselves. In general: For an n-folding, such that $n = k!$, there are $k(k-1)$ pairs of automorphisms in the n-folding such that for every class of substitution variants for unary predicates, there are exactly two sentences that are mapped to each other by the two automorphisms in these pairs and all other sentences in the class of substitution variants are mapped to themselves. I hypothesize that this is not only necessary but also a sufficient criterion for a sentence to be a binary predicate followed by two singular terms.

- In fact, there is something more general to notice here. For every class of substitution variants that is generated by a unary predicate, the set of all automorphisms that swap just two singular terms are a basis of our n-folding. And we can find this basis merely by looking at our symmetries. To find it, we must find an automorphism and its inverse for every pair in our substitution class such that this pair of automorphisms swaps just two sentences in the class and maps all other sentences in the class to themselves. And the set of these automorphisms for every pair in the class must be a basis of our n-folding.
- Moreover, we should find the same basis of our n-folding for every class of substitution variants that correspond to a uniary predicate. I am not sure what we should say if different classes that look like they are classes of unary predicates yield different basis automorphisms.
- Note also that we know the cardinality of classes of substitution variants of uniary predicates. It is the cardinality of our singular terms, and this is k for an n-folding such that $n = k!$.
- For binary predicates, we expect classes of subtitution variants of cardinality k^2 for an n-folding such that $n = k!$.

- To see why a basis like that cannot be found in substitution classes of binary predicates, consider what pattern swaps of singular terms lead to in such classes. Consider a sequence of three binary swaps: $(abc) \mapsto (bac) \mapsto (cab) \mapsto (acb)$; which gets us back to where we started when we repeat it: $(acb) \mapsto (bca) \mapsto (cba) \mapsto (abc)$. And the pattern this produces in uniary predicates is this:
  * First round:
  * $\frac{F(a)}{F(c)\ \ F(b)} \mapsto \frac{F(b)}{F(c)\ \ F(a)} \mapsto \frac{F(c)}{F(b)\ \ F(a)} \mapsto \frac{F(a)}{F(b)\ \ F(c)}$
  * Second round:
  * $\frac{F(a)}{F(b)\ \ F(c)} \mapsto \frac{F(b)}{F(a)\ \ F(c)} \mapsto \frac{F(c)}{F(a)\ \ F(b)} \mapsto \frac{F(a)}{F(c)\ \ F(b)}$
  * And no other sentences in the class are affected (they are all mapped to themselves)
  * Call this the "triple swap dance of unary predicates".
- Now take the class for $R(\cdot,\cdot)$, and take again our sequence of three swaps. The first thing to notice is that a much larger number of sentences in the class are affected by each swap, namely $4k-4$ many each time.
- For $d \neq a, b, c$, we will find that $R(d,a)$, $R(d,b)$, $R(d,c)$, $R(a,d)$, $R(b,d)$, $R(c,d)$ will do just do the triple swap dance of unary predicates when you perform two rounds of binary swaps from above again. The same goes for the instances with double occurrences of a, b, c.
- However, if you do the two rounds of binary swaps from above again, you will find that a different pattern emerges for

sentences in which both arguments are one of a, b, c.

	$R(a,b)$	$R(a,c)$	$R(b,a)$	$R(b,c)$	$R(c,a)$	$R(c,b)$
$a \leftrightarrow b$	$R(b,a)$	$R(b,c)$	$R(a,b)$	$R(a,c)$	$R(c,b)$	$R(c,a)$
$b \leftrightarrow c$	$R(c,a)$	$R(c,b)$	$R(a,c)$	$R(a,b)$	$R(b,c)$	$R(b,a)$
$c \leftrightarrow a$	$R(a,c)$	$R(a,b)$	$R(c,a)$	$R(c,b)$	$R(b,a)$	$R(b,c)$
$a \leftrightarrow b$	$R(b,c)$	$R(b,a)$	$R(c,b)$	$R(c,a)$	$R(a,b)$	$R(a,c)$
$b \leftrightarrow c$	$R(c,b)$	$R(c,a)$	$R(b,c)$	$R(b,a)$	$R(a,c)$	$R(a,b)$
$c \leftrightarrow a$	$R(a,b)$	$R(a,c)$	$R(b,a)$	$R(b,c)$	$R(c,a)$	$R(c,b)$

- Here there are six sentences that are mapped always to one another. Our two rounds of binary-swaps make them dance around in a circle once. Of course, that much is also true of our unary sentences. But for the unary sentences it is always groups of three that dance around in a circle, while here every sentence dances to the spot of all of the five other sentences.

- Call this the "triple swap dance of binary predicates / relations".

- Result: In classes of substitution variants that correspond to relational predicates, when we take three automorphisms that are in a basis of our n-folding, as discovered by looking at unary predicates, and do two rounds of these automorphisms that get us back to the starting point, then we always find exactly one group of six sentences that perform the triple swap dance of binary predicates, while all other sentences perform the triple swap dance of unary predicates.

- It seems to me that there is a characteristic cardinality and a characteristic swap dance for each arity of predicates. If that is true, then we can identify the arity of our predicates by looking at the movements of sentences when we do some series of basis automorphisms.

- Ad 2 above: Can we determine which sentences share occurrences of singular terms?
  - Sentences that are always mapped to themselves by every automorphism in an n-folding don't contain singular terms. So they cannot share any singular terms with any other sentence.
  - All other sentence share are least one of two singular terms iff they are not mapped onto themselves by the same basis automorphism, as determined by the class of unary predicates.
  - To see this, recall that basis automorphisms are swaps of single pairs of singular terms.
  - We can define a singular term as what is shared by sentences such that there is exactly one such sentence in every class of unary predicates and these sentences in their respective classes are that all and only not mapped to themselves by the same set of basis automorphisms. E.g., the swaps (a, b) and (a, c) and (a, d) etc. is the set of automorphisms such that for every unary predicate $F_i(\cdot)$, they map $F_i(a)$ to something other than itself.
  - So let a singular term be the intersection of the nontrivial parts of the domains (i.e. the one's that are not the identity mapping) of two basis automorphisms whose nontrivial domains have a non-empty intersection. This immediately covers predicates of higher arity because sentences in the classes of these predicates can be in the non-trivial parts of the domains of basis automorphisms.

6 Final Remarks

Notice that all of this has a metaphysical as well as a logico-syntactic reading. Here is how we might put the metaphysical readings into some slogans.

- What it means for the world to have object-property structure is that the structure of compatibilities between states has certain kinds of symmetries.
- Objects must be colorless because their permutations define the axes of symmetry of the modal structure of the world.
- Objects are what states that are nontrivially affects by the most basic kind of symmetries have in common.
- Properties are what shared among states that are mapped to each other by the symmetries of the modal structure of the world.
- What it is for a state to include a relation is for some symmetries map the state to states that perform the triple swap dance of relations upon rotation through same basic symmetries.

This is all very abstract and rough. But I am optimistic that this can be made to work.